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Convolutions of singular measures and applications to the Zakharov system

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Abstract

Uniform L^2-estimates for the convolution of singular measures with respect to transversal submanifolds are proved in arbitrary space dimension. The results of Bennett–Bez are used to extend previous work of Bejenaru–Herr–Tataru. As an application, it is shown that the 3D Zakharov system is locally well-posed in the full subcritical regime.

Keywords: Convolution estimates; Transversal submanifolds; L^2 bounds; Zakharov system; Well-posedness

1. Introduction and main results

In this paper we complete the development of a geometric multilinear L^2-estimate which streamlines the analysis of a general class of bilinear forms which appear in various types of nonlinear PDE. In [2] Tataru and the authors proved uniform estimates for the convolution of L^2 measures supported on transversal surfaces in three dimensions. These estimates were close to

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results obtained in [4] by Bennett, Carbery and Wright. In the present paper we generalize our previous result to higher dimensions by using the recent work [3] of Bennett–Bez.

As an application, we establish a sharp result for the Zakharov system in 3D. Our result, when combined with the results in [1,8], closes the full subcritical regime (in the sense of [8, p. 387]) for the Zakharov system in all dimensions. As a consequence, the remaining part of the paper is organized in two sections, each containing results of independent interest.

1.1. Convolutions of singular measures

The first part of the paper is dedicated to a generalization to higher dimensions of the results in [2]. We consider three subsets Σ_1, Σ_2, Σ_3 of submanifolds of $\mathbb{R}^n$ whose codimensions add up to n and which are transversal in the sense that the normal spaces at each point span $\mathbb{R}^n$ and which satisfy certain regularity assumptions. In this set-up we study the restriction to Σ_3 of the convolution of two measures supported on Σ_1, Σ_2. Our main results are global L^2 estimates.

We rely on the result on nonlinear Brascamp–Lieb inequalities proved in [3], see also [4]. More precisely, we utilize the $m = 3$ case of [3, Theorem 1.3] in order to extend the trilinear case of [3, Theorem 7.1] to submanifolds of general codimensions, formulated under global, quantitative assumptions in the spirit of [2].

Before we formulate the precise assumptions on the submanifolds, let us introduce some notations. For given $m_1, m_2, m_3 \in \mathbb{N}$ we define the sets of indices $M_1 = \{1, \dots, m_1\}$, $M_2 = \{m_1 + 1, \dots, m_1 + m_2\}$, and $M_3 = \{m_1 + m_2 + 1, \dots, m_1 + m_2 + m_3\}$. Moreover, for a function $\phi : U \subset \mathbb{R}^{n-m} \to \mathbb{R}^m$ we write $\operatorname{graph}(\phi) = \{(x, \phi(x))^t \in \mathbb{R}^n \colon x \in U\}$.

Assumption 1.1. There exist $0 < \beta \leqslant 1$, $b > 0$, $\theta > 0$, $R > 0$, and $m_i \in \mathbb{N}$ for $i = 1, 2, 3$, with $n = m_1 + m_2 + m_3$, such that

(i) for every $i = 1, 2, 3$ there exists an open set $U_i \subset \mathbb{R}^{n_i}$, $n_i = n - m_i$, and $\phi_i \in C^{1,\beta}(U_i; \mathbb{R}^{m_i})$ with the property

$$R^{-\beta} \sup_{x \in U_i} \bigl|D\phi_i(x)\bigr| + \sup_{x, x' \in U_i} \frac{|D\phi_i(x) - D\phi_i(x')|}{|x - x'|^\beta} \leqslant b, \tag{1.1}$$

such that Σ_i is relatively open, and compactly contained in $G_i \operatorname{graph}(\phi_i)$ for some orthogonal transformation $G_i \in O(n)$;

(ii) for every $i = 1, 2, 3$ and $\sigma_i \in \Sigma_i$ and any orthonormal basis $\{\mathfrak{n}_k\}_{k \in M_i}$ of the normal space $N_{\sigma_i}(\Sigma_i)$ the determinant

$$d(\sigma_1, \sigma_2, \sigma_3) = \det\bigl(\mathfrak{n}_1(\sigma_1), \dots, \mathfrak{n}_{m_1}(\sigma_1), \dots, \mathfrak{n}_{m_1+m_2+1}(\sigma_3), \dots, \mathfrak{n}_n(\sigma_3)\bigr)$$

satisfies the uniform transversality condition

$$\inf_{\sigma_1, \sigma_2, \sigma_3} \bigl|d(\sigma_1, \sigma_2, \sigma_3)\bigr| = \theta; \tag{1.2}$$

(iii) for every $i = 1, 2, 3$ it holds

$$\operatorname{diam}(\Sigma_i) \leqslant R. \tag{1.3}$$

Remark 1. The quantity $d(\sigma_1, \sigma_2, \sigma_3)$ is invariant under changes of the orthonormal bases within each normal space.

Remark 2. Our present setup slightly differs from the one in [2, Assumption 1.1], mainly because here we choose to write the assumptions in terms of graph representation of the involved surfaces.

We identify $f \in L^2(\Sigma_i) = L^2(\Sigma_i, \mu_i)$ – μ_i being the n_i-dimensional Hausdorff-measure – with the distribution

$$\langle f, \psi \rangle = \int_{\Sigma_i} f(y)\psi(y)\, d\mu_i(y), \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

For $f \in L^2(\Sigma_1)$, $g \in L^2(\Sigma_2)$ with compact support the convolution $f * g$ is defined as the distribution

$$\langle f * g, \psi \rangle = \int_{\Sigma_1} \int_{\Sigma_2} f(x)g(y)\psi(x+y)\, d\mu_1(x)\, d\mu_2(y), \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

Since a-priori the restriction of $f * g$ to sets of measure zero is not well defined, we begin with $f \in C_0(\Sigma_1)$ and $g \in C_0(\Sigma_2)$. Then $f * g \in C_0(\mathbb{R}^n)$ and has a well-defined trace on Σ_3. Once we have proved an appropriate L^2-bound, the trace of $f * g$ on Σ_3 can be defined by density for arbitrary $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$.

Following the ideas of [2] we first note the behavior under linear transformations.

Proposition 1.2. *Let Σ_1, Σ_2, Σ_3 satisfy the Assumption* 1.1 *with*

$$\theta \leqslant \left|d(\sigma_1, \sigma_2, \sigma_3)\right| \leqslant 2\theta,$$

and suppose that the estimate

$$\|f * g\|_{L^2(\Sigma_3)} \leqslant C\theta^{-\frac{1}{2}} \|f\|_{L^2(\Sigma_1)} \|g\|_{L^2(\Sigma_2)}, \tag{1.4}$$

holds true for all functions $f \in L^2(\Sigma_1)$, $g \in L^2(\Sigma_2)$. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible, linear map and $\Sigma_i' = T\Sigma_i$, then the estimate

$$\left\|f' * g'\right\|_{L^2(\Sigma_3')} \leqslant 2C\theta'^{-\frac{1}{2}} \left\|f'\right\|_{L^2(\Sigma_1')} \left\|g'\right\|_{L^2(\Sigma_2')} \tag{1.5}$$

holds true for all functions $f' \in L^2(\Sigma_1')$, $g' \in L^2(\Sigma_2')$, where

$$\theta' = \inf_{\sigma_1', \sigma_2', \sigma_3'} \left|d'\left(\sigma_1', \sigma_2', \sigma_3'\right)\right|$$

is defined in analogy to Assumption 1.1(ii).

In summary, the size of the constant is determined only by the transversality properties of the submanifolds.

Next, we look at the fully transversal case. The dual formulation of a local version of the following result for codimension 1 submanifolds is contained in [3, Theorem 7.1].

Theorem 1.3. *Let Σ_1, Σ_2, Σ_3 be submanifolds in $\mathbb{R}^n$ which satisfy Assumption* 1.1 *with parameters $0 < \beta \leqslant 1$, $b = 1$ and $\theta = \frac{1}{2}$, and $R = 1$. Then for each $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$ the restriction of the convolution $f * g$ to Σ_3 is a well-defined $L^2(\Sigma_3)$-function which satisfies*

$$\|f * g\|_{L^2(\Sigma_3)} \leqslant C\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}, \tag{1.6}$$

where the constant C depends only on β and n.

In Section 2 this result will be derived as a consequence of [3, Theorem 1.3]. Finally, in view of future applications, we are interested in how the estimate depends on the more general hypothesis of Assumption 1.1.

Corollary 1.4. *Let Σ_1, Σ_2, Σ_3 be submanifolds in $\mathbb{R}^n$ which satisfy Assumption* 1.1 *with parameters $0 < \beta \leqslant 1$, $b > 0$, $0 < \theta \leqslant 1/2$. Then for each $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$ the restriction of the convolution $f * g$ to Σ_3 is a well-defined $L^2(\Sigma_3)$-function which satisfies*

$$\|f * g\|_{L^2(\Sigma_3)} \leqslant C\theta^{-\frac{1}{2}}\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}, \tag{1.7}$$

where C depends only on β, n, and the size of the quantity $R^\beta b\theta^{-1}$.

1.2. The 3D Zakharov system

In this section we consider the initial value problem associated with the Zakharov system

$$\begin{gathered} i\partial_t u + \Delta u = nu \quad \text{in } (0,T)\times\mathbb{R}^3, \\ \partial_t^2 n - \Delta n = \Delta|u|^2 \quad \text{in } (0,T)\times\mathbb{R}^3, \\ (u, n, \partial_t n)|_{t=0} \in H^s\big(\mathbb{R}^3\big)\times H^\sigma\big(\mathbb{R}^3\big)\times H^{\sigma-1}\big(\mathbb{R}^3\big). \end{gathered} \tag{1.8}$$

The Zakharov system is a model for Langmuir oscillations in a plasma, cf. [12] and [11, Chapter 13] for more information.

Local weak solutions for (1.8) with smooth data were constructed by Sulem and Sulem in [10], and local well-posedness for data in $H^2 \times H^1 \times L^2$ was established by Ozawa and Tsutsumi in [9]. Provided that the Schrödinger part is small in H^1, global well-posedness for data in the energy space, see [6] for details, was established by Bourgain and Colliander in [6].

We are interested in the low regularity well-posedness theory of (1.8). Our notion of well-posedness includes existence of generalized solutions, uniqueness in a suitable subspace, local Lipschitz continuity and persistence of initial regularity. It has been shown by Ginibre, Tsutsumi and Velo in [8] that (1.8) is locally well-posed for $\sigma \geqslant 0$, $2s \geqslant \sigma + 1$, $\sigma \leqslant s \leqslant \sigma + 1$. We extend this result to the full subcritical range in the sense of [8, p. 387].

Theorem 1.5. *The Cauchy problem* (1.8) *is locally well-posed in* $H^s(\mathbb{R}^3) \times H^\sigma(\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3)$ *for* $\sigma > -\frac{1}{2}$, $\sigma \leqslant s \leqslant \sigma + 1$, $2s > \sigma + \frac{1}{2}$.

For a more detailed statement we refer the reader to [1, Theorem 1.1].

The almost admissible endpoint $(s, \sigma) = (0, -\frac{1}{2})$, i.e. bottom left corner of the convex region of admissible (s, σ), matches the 2D result obtained in [1, Theorem 1.1] and extends the result of [8, formula (1.10)] for dimensions $d \geqslant 4$ to $d = 3$.

2. Convolution estimates

Proof of Proposition 1.2. By density and duality, the claimed estimate is equivalent to

$$I(f,g,h) := \int f\big(\sigma_1'\big) g\big(\sigma_2'\big) h\big(\sigma_3'\big) \delta\big(\sigma_1' + \sigma_2' - \sigma_3'\big)\, d\mu_1'\big(\sigma_1'\big)\, d\mu_2'\big(\sigma_2'\big)\, d\mu_3'\big(\sigma_3'\big)$$
$$\leqslant 2C\theta'^{-\frac{1}{2}} \|f\|_{L^2(\Sigma_1')} \|g\|_{L^2(\Sigma_2')} \|h\|_{L^2(\Sigma_3')}, \tag{2.1}$$

for all non-negative, continuous f, g, h. We assume that $\varphi_i : \Omega_i \subset \mathbb{R}^{n_i} \to \mathbb{R}^n$ is a global parametrization for Σ_i, $i = 1, 2, 3$. In that case $\varphi_i' := T\varphi_i$ is a parametrization for Σ_i', $i = 1, 2, 3$. With this parameterizations we have

$$I(f,g,h) = \int f\big(\varphi_1'(x)\big) g\big(\varphi_2'(y)\big) h\big(\varphi_3'(z)\big) \big(g_1' g_2' g_3'\big)^{\frac{1}{2}}\, d\nu'(x,y,z)$$

where $g_i' = \det(D\varphi_i')^t D\varphi_i'$, and with respect to the measure

$$d\nu'(x,y,z) = \delta\big(\varphi_1'(x) + \varphi_2'(y) - \varphi_3'(z)\big)\, dx\, dy\, dz.$$

With $g_i = \det(D\varphi_i)^t D\varphi_i$ and the measure

$$d\nu(x,y,z) = \delta\big(\varphi_1(x) + \varphi_2(y) - \varphi_3(z)\big)(g_1 g_2 g_3)^{\frac{1}{2}}\, dx\, dy\, dz$$

an upper bound on $I(f,g,h)$ is given by

$$\frac{\sup M(x,y,z)}{|\det T|} \int \tilde{f}\big(\varphi_1(x)\big) \tilde{g}\big(\varphi_2(y)\big) \tilde{h}\big(\varphi_3(z)\big) (g_1 g_2 g_3)^{\frac{1}{2}}\, d\nu(x,y,z)$$
$$\leqslant 2\theta^{\frac{1}{2}} \theta'^{-\frac{1}{2}} \int \tilde{f}(\sigma_1) \tilde{g}(\sigma_2) \tilde{h}(\sigma_3) \delta(\sigma_1 + \sigma_2 - \sigma_3)\, d\mu_1(\sigma_1)\, d\mu_2(\sigma_2)\, d\mu_3(\sigma_3)$$
$$\leqslant 2C\theta'^{-\frac{1}{2}} \|f\|_{L^2(\Sigma_1')} \|g\|_{L^2(\Sigma_2')} \|h\|_{L^2(\Sigma_3')},$$

where we have used the definitions

$$M = \prod_{i=1}^{3} g_i'^{\frac{1}{4}} g_i^{-\frac{1}{4}}, \qquad \tilde{f} = g_1'^{\frac{1}{4}} g_1^{-\frac{1}{4}} f(T\cdot),$$

similarly for $\tilde{g}, \tilde{h}$. We have also used that Dirac's δ obeys the simple rule

$$\delta\big(T\varphi_1(x) + T\varphi_2(y) - T\varphi_3(z)\big) = (\det T)^{-1} \delta\big(\varphi_1(x) + \varphi_2(y) - \varphi_3(z)\big)$$

and the following identity

$$\frac{M(x,y,z)}{\det T} = \left(\frac{d(\varphi_1(x),\varphi_2(y),\varphi_3(z))}{d'(\varphi_1'(x),\varphi_2'(y),\varphi_3'(z))}\right)^{\frac{1}{2}}, \tag{2.2}$$

so that (2.2) is the only claim which remains to be proved.

For brevity, let $\sigma_1 = \varphi_1(x)$, $\sigma_2 = \varphi_2(y)$, $\sigma_3 = \varphi_3(z)$, $\sigma_i' = T\sigma_i$, be arbitrary points on Σ_i, which will be fixed for the subsequent calculation.

For $i = 1, 2, 3$ we fix orthonormal bases $\{\mathfrak{n}_k(\sigma_i)\}_{k\in M_i}$ of the normal spaces and define the invertible matrix

$$S = S(\sigma_1,\sigma_2,\sigma_3) = \big(\mathfrak{n}_1(\sigma_1),\dots,\mathfrak{n}_{m_1}(\sigma_1),\dots,\mathfrak{n}_{n_3+1}(\sigma_3),\dots,\mathfrak{n}_n(\sigma_3)\big)^t$$

as well as $R = R(\sigma_1,\sigma_2,\sigma_3) = TS^{-1}$. Then, $T = RS$ and S has the property that if $\Sigma_i'' = S\Sigma_i$ then $\{\mathfrak{e}_k\}_{k\in M_i}$ is an orthonormal basis of the normal space of Σ_i'' at $S\sigma_i$, $i = 1, 2, 3$. We observe that

$$\frac{\det((TD\varphi_i)^t TD\varphi_i)}{\det((D\varphi_i)^t D\varphi_i)} = \frac{\det((SD\varphi_i)^t SD\varphi_i)}{\det((D\varphi_i)^t D\varphi_i)} \cdot \frac{\det((RSD\varphi_i)^t RSD\varphi_i)}{\det((SD\varphi_i)^t SD\varphi_i)}. \tag{2.3}$$

Thus, without restricting the generality of the problem, we can assume that an orthonormal basis of the normal space of Σ_i' at $\sigma_i' = T\sigma_i$ is given as $\{\mathfrak{e}_k\}_{k\in M_i}$, since this takes care of the first factor and it also provides the computation for the reverse situation which takes care of the second factor.

Under this assumption the rows n_k^t of T, i.e. $n_k := T^t\mathfrak{e}_k$, $k\in M_i$ form a basis of the normal space of Σ_i at σ_i, but not necessarily an orthonormal basis. We rely on two basic geometric facts. The first is that $(\det(A^tA))^{\frac{1}{2}}$ is the p-dimensional volume of the parallelepiped spanned by the columns of $A\in\mathbb{R}^{n\times p}$. The second is that if

$$A = (A_1 \mid A_2), \qquad A_k \in \mathbb{R}^{n\times p_k}, \quad p_1 + p_2 = n, \text{ and } \mathrm{R}(A_1)\perp \mathrm{R}(A_2),$$

then the volume of the parallelepiped spanned by the columns of A is the product of the volumes of the parallelepipeds spanned by the columns of A_1, A_2, respectively, i.e.

$$\det(A) = \big(\det\big(A_1^tA_1\big)\big)^{\frac{1}{2}}\big(\det\big(A_2^tA_2\big)\big)^{\frac{1}{2}}.$$

We define the submatrices

$$N_i' = (\mathfrak{e}_{k_i+1},\dots,\mathfrak{e}_{k_i+m_i}), \quad \text{with } k_i \text{ such that } M_i = \{k_i+1,\dots,k_i+m_i\},$$

and $N_i = T^tN_i'$, where the columns n_k are normal to Σ_i, but do not necessarily form an orthonormal set. We compute for $i = 1, 2, 3$ based on the considerations above that

$$\begin{aligned}\frac{\det((TD\varphi_i)^t TD\varphi_i)}{\det((D\varphi_i)^t D\varphi_i)} &= \frac{\det(TD\varphi_i)^t(TD\varphi_i)\det(N_i^tN_i)}{\det^2(D\varphi_i \mid N_i)}\\ &= \frac{\det^2(T)\det(TD\varphi_i)^t(TD\varphi_i)\det(N_i^tN_i)}{\det^2(TD\varphi_i \mid TN_i)}\end{aligned}$$

where here in the sequel we suppress the evaluation of φ_i at x, y, z, respectively. Next, we use

$$\det(TD\varphi_i \mid TN_i) = \det(TD\varphi_i \mid P_i TN_i),$$

where P_i is the orthogonal projection onto N_i', and conclude

$$\det(TD\varphi_i \mid TN_i) = \bigl(\det\bigl((P_iTN_i)^t P_iTN_i\bigr)\bigr)^{\frac{1}{2}}\bigl(\det\bigl((TD\varphi_i)^t TD\varphi_i\bigr)\bigr)^{\frac{1}{2}},$$

such that in summary

$$\frac{\det((TD\varphi_i)^t TD\varphi_i)}{\det((D\varphi_i)^t D\varphi_i)} = \frac{\det^2(T)\det(N_i^t N_i)}{\det((P_iTN_i)^t P_iTN_i)} = \frac{\det^2 T}{\det(N_i^t N_i)}.$$

In the last step we have used the particular form of the vectors in N_i' and the fact that $T^t = (N_1 \mid N_2 \mid N_3)$, which yields

$$\det\bigl((P_iTN_i)^t P_iTN_i\bigr) = \det^2\bigl(N_i^t N_i\bigr).$$

The above computation holds for all $i \in \{1, 2, 3\}$, therefore

$$\frac{M(x,y,z)}{\det T} = (\det T)^{-1}\prod_{i=1}^{3}\left(\frac{\det^2 T}{\det(N_i^t N_i)}\right)^{\frac{1}{4}} = \left(\frac{\det(N_1 \mid N_2 \mid N_3)}{\prod_{i=1}^{3}(\det(N_i^t N_i))^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

This expression is invariant with respect to the choice of normal vectors in N_i, hence we can use an orthonormal set to obtain

$$\frac{M(x,y,z)}{\det T} = \bigl(d\bigl(\varphi_1(x), \varphi_2(y), \varphi_3(z)\bigr)\bigr)^{\frac{1}{2}}$$

and in view of our previous reduction in (2.3) the claim (2.2) follows. This ends the proof of Proposition 1.2. □

Proof of Theorem 1.3. In what follows we use Landau's notation $o(1)$ for scalars, vectors or matrices to denote a quantity which can be made arbitrarily small as $R = \max(\operatorname{diam}(\Sigma_1), \operatorname{diam}(\Sigma_2), \operatorname{diam}(\Sigma_3)) \to 0$. For brevity we introduce the shorthand notation

$$(x_i, \dots, x_j)^t = x_{i,j}, \quad i < j.$$

We subdivide the proof into two steps:

Step 1. By a finite partition (depending only on the dimension), linear changes of coordinates as in the proof of Corollary 1.4 below we can reduce the problem to the following set-up: there exists a triplet $(\sigma_1^0, \sigma_2^0, \sigma_3^0) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$ where $\{\mathfrak{e}_k\}_{k \in M_i}$ is a basis for $N_{\sigma_i^0}(\Sigma_i)$, $i = 1, 2, 3$, such that by the implicit function theorem we have $C^{1,\beta}$-parametrizations $\varphi_i : \Omega_i \to \mathbb{R}^n$, for open subsets Ω_i of the unit ball in $\mathbb{R}^{n_i}$, centered at a_i^0, given as

$$\begin{aligned}
\varphi_1(x_{m_1+1,n}) &= \big(\gamma_1(x_{m_1+1,n}), \dots, \gamma_{m_1}(x_{m_1+1,n}), x_{m_1+1,n}\big)^t,\\
\varphi_2(x_{1,m_1}, x_{m_1+m_2+1,n}) &= \big(x_{1,m_1}, \gamma_{m_1+1}(x_{1,m_1}, x_{m_1+m_2+1,n}), \dots,\\
&\qquad \gamma_{m_1+m_2}(x_{1,m_1}, x_{m_1+m_2+1,n}), x_{m_1+m_2+1,n}\big)^t,\\
\varphi_3(x_{1,n_3}) &= \big(x_{1,n_3}, \gamma_{n_3+1}(x_{1,n_3}), \dots, \gamma_n(x_{1,n_3})\big)^t,
\end{aligned}$$

where $m_i = n - n_i$, such that $\Sigma_i = \varphi_i(\Omega_i)$, $\operatorname{diam}(\Sigma_i)$ is small enough, $\varphi_i(a_i^0) = \sigma_i^0$, where the submanifolds intersect $\varphi_1(a_1^0) + \varphi_2(a_2^0) = \varphi_3(a_3^0)$, and

$$\partial_l \gamma_j\big(a_i^0\big) = 0, \quad \text{for all } j \in M_i, \text{ and all } 1 \leqslant l \leqslant n_i. \tag{2.4}$$

Step 2. We have that for each $i \in \{1, 2, 3\}$

$$\det\big[D\varphi_i^t D\varphi_i\big]\big(a_i^0\big) = 1, \qquad \det\big[D\varphi_i^t D\varphi_i\big] = 1 + o(1) \tag{2.5}$$

and the determinant of the normals satisfies

$$d\big(\sigma_1^0, \sigma_2^0, \sigma_3^0\big) = 1, \qquad d(\sigma_1, \sigma_2, \sigma_3) = 1 + o(1).$$

In this set-up, we need to estimate

$$\begin{aligned}
&\int (f \circ \varphi_1)(x_{m_1+1,n})(g \circ \varphi_2)(x_{1,m_1}, y_{m_1+m_2+1,n})(h \circ \varphi_3)(y_{1,n_3})\\
&\quad \delta\big(\varphi_1(x_{m_1+1,n}) + \varphi_2(x_{1,m_1}, y_{m_1+m_2+1,n}) - \varphi_3(y_{1,n_3})\big)\\
&\quad \big(\det\big[D\varphi_1^t D\varphi_1\big] \det\big[D\varphi_2^t D\varphi_2\big] \det\big[D\varphi_3^t D\varphi_3\big]\big)^{\frac{1}{2}}\, dx_{1,n}\, dy_{1,n}.
\end{aligned}$$

For the function

$$F(x_{1,n}, y_{1,n}) = \varphi_1(x_{m_1+1,n}) + \varphi_2(x_{1,m_1}, y_{m_1+m_2+1,n}) - \varphi_3(y_{1,n_3})$$

it follows from the implicit function theorem that there exists a $C^{1,\beta}$ function G such that $F(x_{1,n}, y_{1,n}) = 0$ if and only if $y_{1,n} = G(x_{1,n})$, since

$$|\det \partial_{y_{1,n}} F| = 1 + o(1), \tag{2.6}$$

because (2.4) yields that the matrix is close to the diagonal matrix with -1 as the first n_3 diagonal entries and $+1$ as the remaining m_3 diagonal entries. Since the following is true

$$\delta\big(F(x_{1,n}, y_{1,n})\big) = |\det \partial_{y_{1,n}} F|^{-1} \delta\big(y_{1,n} - G(x_{1,n})\big),$$

the above integral the above integral can be rewritten as

$$\begin{aligned}
&\int (f \circ \varphi_1)(x_{m_1+1,n})(g \circ \varphi_2)\big(x_{1,m_1}, G_{m_1+m_2+1,n}(x_{1,n})\big)\\
&\quad (h \circ \varphi_3)\big(G_{1,n_3}(x_{1,n})\big) m(x_{1,n})\, dx_{1,n}
\end{aligned}$$

where $m(x_{1,n}) = 1 + o(1)$ in the domain of integration, which follows from (2.5) and (2.6). Then, following the ideas in [3,4], we define the maps $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ by

$$B_1 x_{1,n} = x_{m_1+1,n}, \qquad B_2 x_{1,n} = \big(x_{1,m_1}, G_{m_1+m_2+1,n}(x_{1,n})\big),$$
$$B_3 x_{1,n} = G_{1,n_3}(x_{1,n}).$$

From the properties of φ_i and (2.5) it follows that B_1, B_2, B_3 are $C^{1,\beta}$ functions. With these notations the above integral becomes

$$\int (f \circ \varphi_1)(B_1 x_{1,n})(g \circ \varphi_2)(B_2 x_{1,n})(h \circ \varphi_3)(B_3 x_{1,n}) m(x_{1,n})\, dx_{1,n}$$

Next, we will verify the assumptions of [3, Theorem 1.3] on the kernels of $DB_i(x_0)$, where $x_0 = ([a_2^0]_{1,m_1}, a_1^0) \in \mathbb{R}^n$. We start with $i = 1$:

$$DB_1(x_0) = \begin{pmatrix} 0 & I_{n_1} \end{pmatrix}$$

hence an orthonormal basis of $\ker DB_1(x_0)$ is of the form $\{\mathfrak{e}_k\}_{k \in M_1}$. For $i = 2$ we compute

$$DG(x_0) = -\big[\big(\partial_{y_{1,n}} F\big(x_0, G(x_0)\big)\big)^{-1} \partial_{x_{1,n}} F(x_0, G(x_0)\big] = \begin{pmatrix} I_{n_3} & 0 \\ 0 & -I_{m_3} \end{pmatrix}$$

which implies

$$DB_2(x_0) = \begin{pmatrix} I_{m_1} & 0 \\ 0 & -I_{m_3} \end{pmatrix}$$

and an orthonormal basis of $\ker DB_2(x_0)$ is of the form $\{\mathfrak{e}_k\}_{k \in M_2}$. Concerning $i = 3$, the computation of $DG(x_0)$ above immediately yields

$$DB_3(x_0) = \begin{pmatrix} I_{n_3} & 0 \end{pmatrix}$$

and an orthonormal basis of $\ker DB_3(x_0)$ is given as $\{\mathfrak{e}_k\}_{k \in M_3}$.

From the above characterizations of the kernels of dB_j, it follows from [3, formula (25)] that

$$\left| \star \bigwedge_{j=1}^{3} \star X_j\big(DB_j(x_0)\big) \right| = 1,$$

where we use the notation of [3]. This allows us to invoke the result of [3, Theorem 1.3] in a small neighborhood of x_0, whose size depends only on β and n. □

For the remaining proof we will follow closely the argument in [2, Proof of Corollary 1.6].

Proof of Corollary 1.4. *Step 1.* We first carry out the proof under the additional hypothesis

$$R^\beta b \theta^{-1} \ll 1. \tag{2.7}$$

We have $\|D\phi_i\| \leqslant bR^\beta \ll 1$ throughout $U_i \subset \mathbb{R}^{n_i}$.

Let $i = 1, 2, 3$ and $\sigma_i^0 \in \Sigma_i$ be fixed. Define the normal vectors $\{n_k(\sigma_i)\}_{k\in M_i}$ at $\sigma_i = G_i \cdot (x, \phi_i(x))^t$ to be the columns of the matrix

$$G_i \begin{pmatrix} -D\phi_i^t(x) \\ I_{m_i} \end{pmatrix} \in \mathbb{R}^{n\times m_i}.$$

These vectors satisfy

$$\left|n_k(\sigma_i) - n_k\left(\sigma_i^0\right)\right| \lesssim bR^\beta \ll \theta, \tag{2.8}$$

for all $k \in M_i$, $i = 1, 2, 3$. By the Gram–Schmidt orthonormalization procedure, we can also construct from $\{n_k(\sigma_i)\}_{k\in M_i}$ an orthonormal basis $\{\mathfrak{n}_k(\sigma_i)\}_{k\in M_i}$ of the normal space at $\sigma_i \in \Sigma_i$ satisfying (2.8), which shows that

$$\left|d\left(\sigma_1^0, \sigma_2^0, \sigma_3^0\right) - d(\sigma_1, \sigma_2, \sigma_3)\right| \ll \theta. \tag{2.9}$$

Moreover, we observe that

$$\left|\left(\sigma_i - \sigma_i^0\right) \cdot n_k\left(\sigma_i^0\right)\right| \lesssim bR^{1+\beta} \ll R\theta, \quad k \in M_i,$$

which shows that Σ_i is contained in a plain layer of thickness $\ll R\theta$ with respect to the $n_k(\sigma_i^0)$ direction, for all $k \in M_i$. For fixed $k \in M_1$ we decompose $\mathbb{R}^n$ into layers of thickness $\ll R\theta$ in the direction of $n_k(\sigma_1^0)$. This subdivides Σ_2 and Σ_3 into pieces contained in such layers. The key observation is that for each piece of Σ_2 there are at most finitely many pieces of Σ_3 which contribute to (1.7) and vice-versa. We run the same scheme for $n_k(\sigma_i^0)$ with $k \in M_i$ and $i = 2, 3$, and the above mentioned almost ℓ^2-orthogonality allows us to reduce desired bound (1.7) to the case when the submanifolds satisfy

$$\left|\left(\sigma_i - \sigma_i^0\right) \cdot n_k\left(\sigma_j^0\right)\right| \ll R\theta, \quad k \in M_j,\ i, j = 1, 2, 3. \tag{2.10}$$

We will apply Proposition 1.2 with the matrix

$$T = R\theta\left(A^t\right)^{-1}, \qquad A = \left(n_1\left(\sigma_1^0\right), \dots, n_{m_1}\left(\sigma_1^0\right), \dots, n_{n_3}\left(\sigma_3^0\right), \dots, n_n\left(\sigma_3^0\right)\right).$$

It remains to show that the submanifolds $\tilde{\Sigma}_i := T^{-1}\Sigma_i$ satisfy the assumptions of Theorem 1.3, i.e.

(i) the size condition $\operatorname{diam}(\tilde{\Sigma}_i) \leqslant 1$,
(ii) the transversality condition (1.2) with $\theta = \frac{1}{2}$,
(iii) the regularity condition (1.1) with $R = b = 1$.

Concerning item (i) we observe that

$$T^{-1}\left(\sigma_i - \sigma_i^0\right) = \frac{1}{R\theta}\left(n_1\left(\sigma_1^0\right) \cdot \left(\sigma_i - \sigma_i^0\right), \dots, n_n\left(\sigma_3^0\right) \cdot \left(\sigma_i - \sigma_i^0\right)\right)^t,$$

such that (2.10) shows $\operatorname{diam}(\tilde{\Sigma}_i) \leqslant 1$.

In order to obtain the transversality condition in (ii), we estimate

$$\left\|A^{-1}\right\| \lesssim |\det A|^{-1} \sim \theta^{-1}, \qquad \|T\| \lesssim R. \tag{2.11}$$

Let $k \in M_i$. We define at $\tilde{\sigma}_i \in \tilde{\Sigma}_i$ a normal vector $\tilde{n}_k(\tilde{\sigma}_i)$ to $\tilde{\Sigma}_i$ by

$$\tilde{n}_k(\tilde{\sigma}_i) = A^{-1} n_k(T\tilde{\sigma}_i). \tag{2.12}$$

By construction for $\tilde{\sigma}_i^0 = T^{-1}\sigma_i^0$ we have $\tilde{n}_k(\tilde{\sigma}_i^0) = \mathfrak{e}_k$. By (2.8) and (2.11) it follows that

$$\left|\tilde{n}_k(\tilde{\sigma}_i) - \mathfrak{e}_k\right| \lesssim R^\beta b \ll \theta. \tag{2.13}$$

Thus, we have found a basis $\{\tilde{n}_k(\tilde{\sigma}_i)\}_{k\in M_i}$ of $N_{\tilde{\sigma}_i}(\tilde{\Sigma}_i)$. By the Gram–Schmidt process, we can recursively construct an orthonormal basis $\{\tilde{\mathfrak{n}}_k(\tilde{\sigma}_i)\}_{k\in M_i}$ with the property (2.13). This in turn yields the desired transversality condition

$$\tilde{d}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3) \geqslant 1/2.$$

Concerning the regularity condition in (iii) we define

$$\tilde{\Phi}_i(p) = (R\theta)^{-1}\left[\left(G_i^{-1}Tp\right)_{n_i+1,n} - \phi_i\left(\left(G_i^{-1}Tp\right)_{1,n_i}\right)\right],$$

such that with $Q_i = T^{-1}G_i(U_i \times \mathbb{R}^{m_i})$ it is

$$\tilde{\Sigma}_i = \left\{p \in Q_i \subset \mathbb{R}^n\colon\ \tilde{\Phi}_i(p) = 0\right\}.$$

We would like to resolve this equation for p_k, $k \in M_i$. Now, for $k \leqslant l$ let $I_{k,l}$ be the $l - k + 1 \times n$ matrix, such that $I_{k,l}p = p_{k,l}$. It is

$$D\tilde{\Phi}_i(p) = I_{n_i+1,n}G_i^t\left(A^t\right)^{-1} - D\phi_i\left(\left(G_i^tTp\right)_{1,n_i}\right)I_{1,n_i}G_i^t\left(A^t\right)^{-1}.$$

To keep the exposition clear we discuss the case $i = 1$ only.

$$D_{1,m_1}\tilde{\Phi}_1(p) = \left(I_{n_1+1,n}G_1^t - D\phi_1\left(\left(G_1^tTp\right)_{1,n_1}\right)I_{1,n_1}G_1^t\right)\left(A^t\right)^{-1}I_{1,m_1}^t.$$

Since $\|D\phi_1\| \ll 1$ in U_1 and by construction of A it holds

$$\left\|I_{n_1+1,n}G_1^t - D\phi_1\left(\left(G_1^tTp\right)_{1,n_1}\right)I_{1,n_1}G_1^t - I_{1,m_1}A^t\right\| \ll 1,$$

which shows that

$$\left\|D_{1,m_1}\tilde{\Phi}_1(p) - I_{m_1}\right\| \ll 1.$$

It also implies that for

$$D_{m_1+1,n}\tilde{\Phi}_1(p) = \left(I_{n_1+1,n}G_1^t - D\phi_1\left(\left(G_1^tTp\right)_{1,n_1}\right)I_{1,n_1}G_1^t\right)\left(A^t\right)^{-1}I_{m_1+1,n}^t$$

we have

$$\big\| D_{m_1+1,n}\tilde{\Phi}_1(p)\big\| \ll 1.$$

At $p=\tilde{\sigma}_1^0$ we evaluate

$$D_{1,m_1}\tilde{\Phi}_1\big(\tilde{\sigma}_1^0\big)=I_{m_1}\quad \text{and}\quad D_{m_1+1,n}\tilde{\Phi}_1\big(\tilde{\sigma}_1^0\big)=0.$$

The implicit function theorem yields a global resolution $\tilde{\phi}_1\in C^{1,\beta}(\tilde{U}_1)$ with domain $\tilde{U}_1=I_{1,n_1}(Q_1)$ such that $\tilde{\Phi}_1(\tilde{\phi}_1(\tilde{x}),\tilde{x})=0$ with $D\tilde{\phi}_1(\tilde{x}^0)=0$ and the analog of (1.1) is satisfied with $R=b=1$.

Step 2. Finally, we remove the additional assumption (2.7). In general we have $R^\beta b\theta^{-1}\gtrsim 1$. We partition each submanifold Σ_i into about $R\delta^{-1}$ pieces of diameter δ for $\delta^\beta b\ll\theta$. It remains to prove that for each such piece we can find a graph representation satisfying Assumption 1.1(i) with R replaced with δ. In order to do so, in each piece we select a point $G_i(a_i^0,\phi_i(a_i^0))^t$ and define a rotation $O_i\in\mathbb{R}^{n\times n}$ with the property

$$O_i\ \text{range}\begin{pmatrix}0\\ I_{m_i}\end{pmatrix}=\text{range}\begin{pmatrix}-D\phi_i^t(a_i^0)\\ I_{m_i}\end{pmatrix},$$

and the implicit function theorem yields a representation of the piece as $G_iO_i\,\text{graph}(\tilde{\phi}_i)$ with vanishing differential at a point. This implies (1.1) with R replaced by δ. $\square$

3. The Zakharov system

3.1. Notation and function spaces

We adopt most of the notations from [1]. We write $A\lesssim B$ if there exists a harmless constant $c>0$ such that $A\leqslant cB$. Moreover, we write $A\gtrsim B$ if $B\lesssim A$ and $A\sim B$ if $A\lesssim B$ and $A\gtrsim B$. Throughout this paper we will denote dyadic numbers 2^n for $n\in\mathbb{N}$ by the corresponding upper-case letters, e.g. $N=2^n$, $L=2^l$, etc.

Let $\psi\in C_0^\infty((-2,2))$ be an even, non-negative function with the property $\psi(r)=1$ for $|r|\leqslant 1$. We use it to define a partition of unity in $\mathbb{R}$,

$$1=\sum_{N\geqslant 1}\psi_N,\quad \psi_1=\psi,\qquad \psi_N(r)=\psi\left(\frac{r}{N}\right)-\psi\left(\frac{2r}{N}\right),\quad N=2^n\geqslant 2.$$

Thus $\operatorname{supp}\psi_1\subset[-2,2]$ and $\operatorname{supp}\psi_N\subset[-2N,-N/2]\cup[N/2,2N]$ for $N\geqslant 2$. For $f:\mathbb{R}^3\to\mathbb{C}$ we define the dyadic frequency localization operators P_N by

$$\mathcal{F}_x(P_Nf)(\xi)=\psi_N\big(|\xi|\big)\mathcal{F}_xf(\xi).$$

For $u:\mathbb{R}^3\times\mathbb{R}\to\mathbb{C}$ we define $(P_Nu)(x,t)=(P_Nu(\cdot,t))(x)$. We will often write $u_N=P_Nu$ for brevity. We denote the space-time Fourier support of P_N by the corresponding Gothic letter

$$\mathfrak{P}_1 = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\xi| \leqslant 2\},$$
$$\mathfrak{P}_N = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid N/2 \leqslant |\xi| \leqslant 2N\}.$$

Moreover, for dyadic $L \geqslant 1$ we define the modulation localization operators

$$\mathcal{F}(S_L u)(\tau,\xi) = \psi_L(\tau + |\xi|^2)\mathcal{F}u(\tau,\xi) \quad \text{(Schrödinger case)}, \tag{3.1}$$
$$\mathcal{F}(W_L^{\pm} u)(\tau,\xi) = \psi_L(\tau \pm |\xi|)\mathcal{F}u(\tau,\xi) \quad \text{(Wave case)}, \tag{3.2}$$

and the corresponding space-time Fourier supports

$$\mathfrak{S}_1 = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau + |\xi|^2| \leqslant 2\},$$
$$\mathfrak{S}_L = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid L/2 \leqslant |\tau + |\xi|^2| \leqslant 2L\},$$

respectively

$$\mathfrak{W}_1^{\pm} = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau \pm |\xi|| \leqslant 2\},$$
$$\mathfrak{W}_L^{\pm} = \{(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid L/2 \leqslant |\tau \pm |\xi|| \leqslant 2L\}.$$

Next we introduce localization operators with respect to angular variables. For each $A \in \mathbb{N}$ we choose a cover $\{\omega_A^j\}_{j\in\Omega_A}$ of $\mathbb{S}^2$ with the following properties:

(i) Each ω_A^j is a spherical cap with angular opening A^{-1}, i.e the angle $\angle(x,y)$ between any two vectors in $x, y \in \omega_A^j$ satisfies

$$|\angle(x,y)| \leqslant A^{-1}.$$

(ii) $\mathbb{S}^2$ is the almost disjoint union of $\{\omega_A^j\}_{j\in\Omega_A}$, i.e. if $\chi_{\omega_A^j}$ denotes the characteristic function of the cap ω_A^j, then have the following

$$1 \leqslant \chi(x) := \sum_{j\in\Omega_A} \chi_{\omega_A^j}(x) \leqslant 3, \quad \forall x \in \mathbb{S}^2,$$

and we require that any two centers of caps in our collection are separated by a distance $\sim A^{-1}$ such that $\#\Omega_A \lesssim A^2$.

Related to this we define the function

$$\alpha(j_1,j_2) = \inf\{|\angle(\pm x, y)|:\ x \in \omega_A^{j_1},\ y \in \omega_A^{j_2}\}$$

which measures the minimal angle between any two straight lines through the caps $\omega_A^{j_1}$ and $\omega_A^{j_2}$, respectively.

Based on the above construction, for each $j \in \Omega_A$ we define

$$\mathfrak{Q}_A^j = \left\{(\xi,\tau) \in \mathbb{R}^3 \Big\backslash \{0\} \times \mathbb{R}\colon \frac{\xi}{|\xi|} \in \omega_A^j\right\},$$

and the corresponding localization operator

$$\mathcal{F}\big(Q_A^j u\big)(\xi,\tau) = \frac{\chi_{\omega_A^j}(\frac{\xi}{|\xi|})}{\chi(\frac{\xi}{|\xi|})}\mathcal{F}u(\xi,\tau).$$

For $k,\ell \in \mathbb{R}$ and $T>0$ we define the space $\mathbf{Z}_T^{k,\ell}$ as the Banach space of all pairs of space-time distributions (u,n) which satisfy

$$\begin{aligned} &u \in C\big([0,T]; H^k\big(\mathbb{R}^3;\mathbb{C}\big)\big),\\ &n \in C\big([0,T]; H^\ell\big(\mathbb{R}^3;\mathbb{R}\big)\big) \cap C^1\big([0,T]; H^{\ell-1}\big(\mathbb{R}^3;\mathbb{R}\big)\big), \end{aligned} \tag{3.3}$$

endowed with the standard norm $\|\cdot\|_{\mathbf{Z}_T^{k,\ell}}$ defined as

$$\big\|(u,n)\big\|_{\mathbf{Z}_T^{k,\ell}}^2 = \sup_{t\in[0,T]}\big\{\big\|u(t)\big\|_{H_x^k}^2 + \big\|n(t)\big\|_{H_x^\ell}^2 + \big\|\partial_t n(t)\big\|_{H_x^{\ell-1}}^2\big\}. \tag{3.4}$$

Let $\sigma, b \in \mathbb{R}$, $1 \leqslant p < \infty$. In connection to the operator $i\partial_t + \Delta$ we define the Bourgain space $X^S_{\sigma,b,p}$ of all $u \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ for which the norm

$$\|u\|_{X^S_{\sigma,b,p}} = \left(\sum_{N\geqslant 1} N^{2\sigma}\left(\sum_{L\geqslant 1} L^{pb}\|S_L P_N u\|_{L^2}^p\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}$$

is finite. Similarly, to the half-wave operators $i\partial_t \pm \langle\nabla\rangle$ we associate the Bourgain spaces $X^{W^\pm}_{\sigma,b,p}$ of all $v \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ for which the norm

$$\|v\|_{X^{W^\pm}_{\sigma,b,p}} = \left(\sum_{N\geqslant 1} N^{2\sigma}\left(\sum_{L\geqslant 1} L^{pb}\big\|W_L^\pm P_N u\big\|_{L^2}^p\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}$$

is finite. For $p=\infty$ we modify the definition as usual. In cases where the Schwartz space $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ is not dense in $X^{W^\pm}_{\sigma,b,p}$ or $X^S_{\sigma,b,p}$, respectively, we redefine the spaces and take the closure of $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ instead.

For a normed space $B \subset \mathcal{S}'(\mathbb{R}^n \times \mathbb{R};\mathbb{C})$ of space-time distributions we denote by $\overline{B}$ the space of complex conjugates with the induced norm.

For $T>0$ we define the space $B(T)$ of restrictions of distributions in B to the set $\mathbb{R}^n \times (0,T)$ with the induced norm

$$\|u\|_{B(T)} = \inf\big\{\|\tilde{u}\|_B\colon \tilde{u} \in B \text{ is an extension of } u \text{ to } \mathbb{R}^n \times \mathbb{R}\big\}.$$

3.2. Multilinear estimates

This section is devoted to the proof of the crucial multilinear estimates in Theorem 3.1 which imply the well-posedness result for the Zakharov system in Theorem 1.5. Given these multilinear estimates, Theorem 1.5 can be deduced by the standard Picard iteration argument as described in [1, Section 5 and Section 3] for the $2d$ case (an alternative argument can be found in [8]). Therefore, in the sequel we will focus on the proof of the following:

Theorem 3.1. *Assume that $s > 0, \sigma > -\frac{1}{2}, \sigma \leqslant s \leqslant \sigma + 1$, $\sigma - 2s < -\frac{1}{2}$.*

(i) *For all $0 < T \leqslant 1$ and for all functions $u, u_1, u_2 \in X^S_{s,\frac{1}{2},1}(T)$ and $v \in X^{W+}_{\sigma,\frac{1}{2},1}(T)$ the following estimates hold true*:

$$\|uv\|_{X^S_{s,-\frac{1}{2},1}(T)} \lesssim \|u\|_{X^S_{s,\frac{1}{2},1}(T)} \|v\|_{X^{W+}_{\sigma,\frac{1}{2},1}(T)}, \tag{3.5}$$

$$\|u\bar{v}\|_{X^S_{s,-\frac{1}{2},1}(T)} \lesssim \|u\|_{X^S_{s,\frac{1}{2},1}(T)} \|v\|_{X^{W+}_{\sigma,\frac{1}{2},1}(T)}, \tag{3.6}$$

$$\left\| \frac{\Delta}{\langle\nabla\rangle}(u_1\bar{u}_2)\right\|_{X^{W+}_{\sigma,-\frac{1}{2},1}(T)} \lesssim \|u_1\|_{X^S_{s,\frac{1}{2},1}(T)} \|u_2\|_{X^S_{s,\frac{1}{2},1}(T)}. \tag{3.7}$$

(ii) *There exists $\theta = \theta(s,\sigma) > 0$ in the above regime for s, σ such that all the inequalities can be improved with a factor of T^θ on the right-hand side.*

We have split the above result in two parts for the following reason. Part (i) contains the "clean" estimates without keeping track of the gains of powers of T which may distract the reader from the main ideas. However, from part (i), we would be able to claim only a small data result for the Zakharov system. It is part (ii) that allows us to claim the local well-posedness result for large data.

We introduce the notation

$$I(f, g_1, g_2) = \int f(\zeta_1 - \zeta_2) g_1(\zeta_1) g_2(\zeta_2)\, d\zeta_1\, d\zeta_2,$$

where $\zeta_i = (\xi_i, \tau_i)$, $i = 1, 2$. Using duality and the fact that $\overline{\mathcal{F}u} = \mathcal{F}\bar{u}(-\cdot)$, we can reduce Theorem 3.1 to the following trilinear estimates:

Proposition 3.2. *Assume that $s > 0$, $\sigma > -\frac{1}{2}$, $\sigma \leqslant s \leqslant \sigma + 1$, $\sigma - 2s < -\frac{1}{2}$.*

(i) *For all $v, u_1, u_2 \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ it holds*

$$\left|I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2)\right| \lesssim \|u_1\|_{X^S_{-s,\frac{1}{2},\infty}} \|u_2\|_{X^S_{s,\frac{1}{2},\infty}} \|v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}}, \tag{3.8}$$

$$\left|I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2)\right| \lesssim \|u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|u_2\|_{X^S_{s,\frac{1}{2},\infty}} \|v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}}. \tag{3.9}$$

(ii) *There exists $b = b(s,\sigma) < \frac{1}{2}$ in the above regime for s,σ such that the above inequalities hold true with $\|u_2\|_{X^S_{s,b,\infty}}$ instead.*

Obviously part (i) in Theorem 3.1 follows from part (i) in the proposition. Part (ii) in Theorem 3.1 follows from part (ii) of the proposition and the following estimate

$$\|f\|_{X_{s,b,1}(T)} \lesssim T^{\frac{1}{2}-b}\|f\|_{X_{s,\frac{1}{2},1}(T)} \tag{3.10}$$

whenever $0 \leqslant b < \frac{1}{2}$. Here $X_{s,b,1}(T)$ stands both for $X^S_{s,b,1}(T)$ and $X^{W\pm}_{s,b,1}(T)$. A proof of (3.10) can be found in [1, Section 5].

The proof of Proposition 3.2 is given at the end of this section. As building blocks we provide a number of preliminary estimates first. These are concerned with functions which are dyadically localized in frequency and modulation. In some cases we additionally differentiate frequencies by their angular separation.

We start this analysis by recalling the well-known bilinear generalization of the linear L^4 Strichartz estimate for the Schrödinger equation in dimension 2 which is essentially due to Bourgain [5, Lemma 111]. We observe that a similar estimate is true for a Wave–Schrödinger interaction.

Proposition 3.3 *(Bilinear Strichartz estimates).*

(i) *Let $u_1, u_2 \in L^2(\mathbb{R}^4)$ be dyadically Fourier-localized such that*

$$\operatorname{supp} \mathcal{F}u_i \subset \mathfrak{P}_{N_i} \cap \mathfrak{S}_{L_i}$$

for $L_1, L_2 \geqslant 1$, $N_1, N_2 \geqslant 1$. Then the following estimate holds:

$$\|u_1 u_2\|_{L^2(\mathbb{R}^4)} \lesssim N_1 N_2^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|u_1\|_{L^2}\|u_2\|_{L^2}. \tag{3.11}$$

(ii) *Let $u, v \in L^2(\mathbb{R}^4)$ be such that*

$$\operatorname{supp} \mathcal{F}v \subset C \times \mathbb{R} \cap \mathfrak{W}^{\pm}_L, \qquad \operatorname{supp} \mathcal{F}u \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}$$

for $L, L_1 \geqslant 1$, $N_1 \geqslant 1$ and a cube $C \subset \mathbb{R}^3$ of sidelength $d \geqslant 1$. Then the following estimate holds:

$$\|uv\|_{L^2(\mathbb{R}^4)} \lesssim \min\{d, N_1\} N_1^{-\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|u\|_{L^2}\|v\|_{L^2}. \tag{3.12}$$

In particular, if

$$\operatorname{supp} \mathcal{F}v \subset \mathfrak{P}_N \cap \mathfrak{W}^{\pm}_L, \qquad \operatorname{supp} \mathcal{F}u \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}$$

for $L, L_1 \geqslant 1$, $N, N_1 \geqslant 1$, it follows

$$\|uv\|_{L^2(\mathbb{R}^4)} \lesssim \min\{N, N_1\} N_1^{-\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|u\|_{L^2}\|v_1\|_{L^2}. \tag{3.13}$$

On the left-hand side of (3.11), (3.12) *and* (3.13) *we may replace each function with its complex conjugate.*

Proof. As remarked above the estimate (3.11) is due to Bourgain [5, Lemma 111] for two dimensions and has been generalized in [7, Lemma 3.4] to higher dimensions.[3] It remains to show (3.12) and (3.13). With $f = \mathcal{F}v$ and $g = \mathcal{F}u$ it follows

$$\left\| \int f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_{\xi,\tau}} \lesssim \sup_{\xi,\tau} \left| E(\xi, \tau) \right|^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}$$

by the Cauchy–Schwarz inequality, where

$$E(\xi, \tau) = \left\{ (\xi_1, \tau_1) \in \operatorname{supp} f \;\middle|\; (\xi - \xi_1, \tau - \tau_1) \in \operatorname{supp} g \right\} \subset \mathbb{R}^4.$$

With $\underline{l} = \min\{L, L_1\}$ and $\bar{l} = \max\{L, L_1\}$ the volume of this set can be estimated as

$$\left| E(\xi, \tau) \right| \leqslant \underline{l} \cdot \left| \left\{ \xi_1 \;\middle|\; \left| \tau \pm |\xi_1| + |\xi - \xi_1|^2 \right| \lesssim \bar{l},\ \xi_1 \in C,\ |\xi - \xi_1| \sim N_1 \right\} \right|,$$

by Fubini's theorem. The latter subset of $\mathbb{R}^3$ is contained in a cube of sidelength m, where $m \sim \min\{d, N_1\}$, so if $N_1 = 1$ the estimate follows. If $N_1 \geqslant 2$ and one component $\xi_{1,i}$, $i \in \{1, 2, 3\}$ is fixed, then the other two components $\xi_{1,j}$, $j \neq i$ are confined to an interval of length m. For each $i \in \{1, 2, 3\}$, we notice that in the subset where $|(\xi - \xi_1)_i| \gtrsim N_1$ we have that $|\partial_{\xi_{1,i}} (\tau \pm |\xi_1| + |\xi - \xi_1|^2)| \gtrsim N_1$. This shows that

$$\left| \left\{ \xi_1 \;\middle|\; \left| \tau \pm |\xi_1| + |\xi - \xi_1|^2 \right| \lesssim \bar{l},\ \xi_1 \in C,\ |\xi - \xi_1| \sim N_1 \right\} \right| \lesssim N_1^{-1} \bar{l} m^2,$$

and the claim (3.12) follows. This also implies the claim (3.13) because the dyadic annulus of radius N is contained in a cube of sidelength $d \sim N$. □

Proposition 3.4 *(Transverse high–high interactions, low modulation). Let $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{Q}_A^{j_k} \cap \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

where the frequencies N, N_1, N_2 and modulations L, L_1, L_2 satisfy

$$1 \ll N \lesssim N_1 \sim N_2, \qquad L_1, L_2, L \lesssim N_1^2$$

while the angular localization parameters A and $j_1, j_2 \in \Omega_A$ satisfy

$$1 \leqslant A \ll N_1, \qquad \alpha(j_1, j_2) \sim A^{-1}.$$

Then the following estimate holds

$$\left| I(f, g_1, g_2) \right| \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}}. \tag{3.14}$$

[3] Note that the proof of [7, Lemma 3.4] applies with $\delta = 0$ for functions which are dyadically Fourier-localized.

Proof. We abuse notation and replace g_2 by $g_2(-\cdot)$ and change variables $\zeta_2 \mapsto -\zeta_2$ to obtain the usual convolution structure. From now on it holds $|\tau_2 - |\xi_2|^2| \sim L_2$ within the support of g_2. We consider only the case $\operatorname{supp}(f) \subset \mathfrak{W}_L^-$ since in the case $\operatorname{supp}(f) \subset \mathfrak{W}_L^+$ the same arguments apply.

For fixed ξ_1, ξ_2 we change variables $c_1 = \tau_1 + |\xi_1|^2$, $c_2 = \tau_2 - |\xi_2|^2$. By decomposing f into $\sim L$ pieces and applying the Cauchy–Schwarz inequality, it suffices to prove

$$\begin{aligned} &\left| \int g_1\big(\varphi_{c_1}^-(\xi_1)\big) g_2\big(\varphi_{c_2}^+(\xi_2)\big) f\big(\varphi_{c_1}^-(\xi_1) + \varphi_{c_2}^+(\xi_2)\big)\, d\xi_1\, d\xi_2 \right| \\ &\quad \lesssim N_1^{-\frac{1}{2}} \big\|g_1 \circ \varphi_{c_1}^-\big\|_{L^2_\xi} \big\|g_2 \circ \varphi_{c_2}^+\big\|_{L^2_\xi} \|f\|_{L^2} \end{aligned} \tag{3.15}$$

where f is now supported in $c \leqslant \tau - |\xi| \leqslant c + 1$ and $\varphi_{c_k}^{\pm}(\xi) = (\xi, \pm|\xi|^2 + c_k)$, $k = 1, 2$, and the implicit constant is independent of c, c_1, c_2.

We refine the localization of the ξ and τ components by orthogonality methods. Since the support of f in the τ direction is confined to an interval of length $\lesssim N_1$, $|\xi_2|^2 - |\xi_1|^2$ is localized in an interval of length $\sim N_1$ which in turn localizes $|\xi_2| - |\xi_1|$ in an interval of size ~ 1. By decomposing the plane into annuli of size ~ 1 and using the Cauchy–Schwarz inequality, we reduce (3.15) further to the additional assumption that $|\xi_1|$ and $|\xi_2|$ are localized in two intervals of length $\sim 1 \lesssim N_1 A^{-1}$. Recalling the additional angular localization, we can assume that g_1, g_2 and f are each localized in cubes of size $N_1 A^{-1}$ with respect to the ξ variables.

We use the parabolic scaling $(\xi, \tau) \mapsto (N_1 \xi, N_1^2 \tau)$ to define

$$\tilde{f}(\xi, \tau) = f\big(N_1 \xi, N_1^2 \tau\big), \qquad \tilde{g}_k(\xi_k, \tau_k) = g_k\big(N_1 \xi_k, N_1^2 \tau_k\big), \quad k = 1, 2.$$

If we set $\tilde{c}_k = c_k N_k^{-2}$, the estimate (3.15) reduces to

$$\begin{aligned} &\left| \int \tilde{g}_1\big(\varphi_{\tilde{c}_1}^-(\xi_1)\big) \tilde{g}_2\big(\varphi_{\tilde{c}_2}^+(\xi_2)\big) \tilde{f}\big(\varphi_{\tilde{c}_1}^-(\xi_1) + \varphi_{\tilde{c}_2}^+(\xi_2)\big)\, d\xi_1\, d\xi_2 \right| \\ &\quad \lesssim N_1^{-1} \big\|\tilde{g}_1 \circ \varphi_{\tilde{c}_1}^-\big\|_{L^2_\xi} \big\|\tilde{g}_2 \circ \varphi_{\tilde{c}_2}^+\big\|_{L^2_\xi} \|\tilde{f}\|_{L^2}, \end{aligned} \tag{3.16}$$

where now $\tilde{g}_k$ is supported in a cube of size $\sim A^{-1}$ with $|\xi_1|, |\xi_2| \sim 1$ and the supports are separated by $\sim A^{-1}$. Note that $\tilde{f}$ is supported in a neighborhood of size N_1^{-2} of the submanifold S_3 parametrized by $\varphi_c^{N_1}(\xi) = (\xi, \frac{|\xi|}{N_1} + \frac{c}{N_1^2})$. Let us put $\varepsilon = N_1^{-2}$ and denote this neighborhood by $S_3(\varepsilon)$. The separation of ξ_1 and ξ_2 above implies also that in the support of $\tilde{f}$ we have $|\xi| \gtrsim A^{-1} \geqslant N_1^{-1}$.

By density and duality it is enough to consider continuous $\tilde{g}_1, \tilde{g}_2$ and we can further rewrite the above estimate as

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3(\varepsilon))} \lesssim \varepsilon^{\frac{1}{2}} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)} \tag{3.17}$$

where S_i, $i = 1, 2$ are parametrized by $\varphi_{\tilde{c}_i}^{\pm}$. The above localization properties of the support of $\tilde{g}_i$ are inherited by S_i, which implies that the maximal diameter of the S_1, S_2 and S_3 is at most $R \sim A^{-1}$.

It is a straightforward to check that S_1, S_2 and S_3 verify part (i) of Assumption 1.1 with $\beta = 1$ and $b \sim 1$.

We now turn our attention to the transversality condition, i.e. part (ii) of Assumption 1.1. Since $\alpha(j_1, j_2) \sim A^{-1}$, there exists a unit vector v which is almost orthogonal to any $\xi_1 \in \mathfrak{Q}_A^{j_1}$ and any $\xi_2 \in \mathfrak{Q}_A^{j_2}$ in the following sense

$$\left|\det\left(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v\right)\right| = \mathrm{vol}\left(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v\right) \sim \left|\sin\angle\left(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}\right)\right| \sim A^{-1}. \tag{3.18}$$

The codimensions of S_1, S_2, S_3 add up to 3 instead of 4. In order to be able to apply the results in the first part of the paper, we foliate one of the surfaces to increase its codimension by one. We do this for S_3 as follows:

$$S_3 = \bigcup_{c \in I} S_3^c$$

where $S_3^c = S_3 \cap \{(\xi, \tau)\colon\ \xi \cdot v = c\}$ and c varies in an interval I of length $|I| \sim A^{-1}$. Each S_3^c retains its $C^{1,1}$ structure. In addition,

$$\|f\|_{L^2(S_3)}^2 = \int_I \|f\|_{L^2(S_3^c)}^2 \, dc \lesssim A^{-1} \sup_c \|f\|_{L^2(S_3^c)}^2. \tag{3.19}$$

For fixed $c \in I$, let us identify a basis of unit normals to S_3^c. For the following calculations, we set $\xi = \xi_0$ and denote the components as

$$\xi_k = (\xi_{k,1}, \xi_{k,2}, \xi_{k,3}), \quad k = 0, 1, 2.$$

At each point we keep the normal to the cone

$$\mathfrak{n}_{S_3} = \left(\frac{\xi_{0,1}}{|\xi|\langle N_1\rangle}, \frac{\xi_{0,2}}{|\xi|\langle N_1\rangle}, \frac{\xi_{0,3}}{|\xi|\langle N_1\rangle}, -\frac{N_1}{\langle N_1\rangle}\right).$$

Another convenient normal is $\mathfrak{n}_{S_3^c} = (v, 0)$. This choice is simple, but it has the disadvantage that $\{\mathfrak{n}_{S_3}, \mathfrak{n}_{S_3^c}\}$ is not an orthonormal basis. On the other hand,

$$|\mathfrak{n}_{S_3} \cdot \mathfrak{n}_{S_3^c}| = \left|v \cdot \left(\frac{\xi_{0,1}}{|\xi|\langle N_1\rangle}, \frac{\xi_{0,2}}{|\xi|\langle N_1\rangle}, \frac{\xi_{0,3}}{|\xi|\langle N_1\rangle}\right)\right| \leqslant \frac{1}{\langle N_1\rangle} \ll 1. \tag{3.20}$$

Therefore, a correct orthonormal set of normals to S_3^c is $\{\mathfrak{n}_{S_3}, \mathfrak{n}'_{S_3}\}$, with

$$\mathfrak{n}'_{S_3} = \frac{\mathfrak{n}_{S_3^c} - (\mathfrak{n}_{S_3} \cdot \mathfrak{n}_{S_3^c})\mathfrak{n}_{S_3}}{|\mathfrak{n}_{S_3^c} - (\mathfrak{n}_{S_3} \cdot \mathfrak{n}_{S_3^c})\mathfrak{n}_{S_3}|}.$$

Now, we can analyze the transversality properties of our submanifolds S_1, S_2, S_3^c in the sense of (1.2). Let $\mathfrak{n}_1, \mathfrak{n}_2$ be the unit normals at S_1, respectively S_2. Then we need to determine the absolute value of the determinant

$$d = \det(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_{S_3}, \mathfrak{n}'_{S_3}) = \frac{1}{|\mathfrak{n}_{S_3^c} - (\mathfrak{n}_{S_3} \cdot \mathfrak{n}_{S_3^c})\mathfrak{n}_{S_3}|} \det(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_{S_3}, \mathfrak{n}_{S_3^c}).$$

In view of (3.20) we obtain

$$d \sim \det(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_{S_3}, \mathfrak{n}_{S_3^c}) = \begin{vmatrix} \frac{2\xi_{1,1}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,1}}{\langle 2\xi_2\rangle} & \frac{\xi_{0,1}}{|\xi_0|\langle N_1\rangle} & v_1 \\ \frac{2\xi_{1,2}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,2}}{\langle 2\xi_2\rangle} & \frac{\xi_{0,2}}{|\xi_0|\langle N_1\rangle} & v_2 \\ \frac{2\xi_{1,3}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,3}}{\langle 2\xi_2\rangle} & \frac{\xi_{0,3}}{|\xi_0|\langle N_1\rangle} & v_3 \\ \frac{1}{\langle 2\xi_1\rangle} & -\frac{1}{\langle 2\xi_2\rangle} & -\frac{N_1}{\langle N_1\rangle} & 0 \end{vmatrix}.$$

Expansion along the third column shows that

$$\left|\det(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_{S_3}, \mathfrak{n}_{S_3^c}) - \tilde{d}\right| \lesssim N_1^{-1},$$

i.e. the main contribution comes from the $(4,3)$-minor

$$\tilde{d} = \frac{N_1}{\langle N_1\rangle} \begin{vmatrix} \frac{2\xi_{1,1}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,1}}{\langle 2\xi_2\rangle} & v_1 \\ \frac{2\xi_{1,2}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,2}}{\langle 2\xi_2\rangle} & v_2 \\ \frac{2\xi_{1,3}}{\langle 2\xi_1\rangle} & \frac{2\xi_{2,3}}{\langle 2\xi_2\rangle} & v_3 \end{vmatrix},$$

which can be rewritten as

$$\tilde{d} = \frac{N_1}{\langle N_1\rangle} \frac{2|\xi_1|}{\langle 2\xi_1\rangle} \frac{2|\xi_2|}{\langle 2\xi_2\rangle} \begin{vmatrix} \frac{\xi_{1,1}}{|\xi_1|} & \frac{\xi_{2,1}}{|\xi_2|} & v_1 \\ \frac{\xi_{1,2}}{|\xi_1|} & \frac{\xi_{2,2}}{|\xi_2|} & v_2 \\ \frac{\xi_{1,3}}{|\xi_1|} & \frac{\xi_{2,3}}{|\xi_2|} & v_3 \end{vmatrix} = \frac{N_1}{\langle N_1\rangle} \frac{2|\xi_1|}{\langle 2\xi_1\rangle} \frac{2|\xi_2|}{\langle 2\xi_2\rangle} \det\left(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v\right)$$

which, by (3.18), implies that $|\tilde{d}| \sim A^{-1} \gg N_1^{-1}$. Therefore we have established that $|d| \sim A^{-1}$. Recalling that the diameters of S_1, S_2, S_3^c are $\sim A^{-1}$, we can now apply Corollary 1.4 which implies

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3^c)} \lesssim A^{\frac{1}{2}} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)}.$$

From (3.19) we obtain

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3)} \lesssim \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)}$$

and (3.17) follows. $\square$

Proposition 3.5 *(Parallel high–high interactions). Let* $f, g_1, g_2 \in L^2$, $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ *such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{Q}_A^{j_k} \cap \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

with $1 \ll N \lesssim N_1 \sim N_2$. *Assume that* $A \sim N_1$ *and* $\alpha(j_1, j_2) \lesssim A^{-1}$. *Then for all* $L, L_1, L_2 \geqslant 1$ *we have*

$$\left|I(f, g_1, g_2)\right| \lesssim N_1^{-\frac{1}{2}}(L_1 L_2 L)^{\frac{1}{2}}. \tag{3.21}$$

Proof. After a rotation we may assume that the angular localization is such that the first spherical cap $\omega_A^{j_1}$ is centered at $(1, 0, 0)$ and the second spherical cap $\omega_A^{j_2}$ is located at distance $\lesssim A^{-1}$ from $(\pm 1, 0, 0)$. Then, if $(\xi_k, \tau_k) \in \operatorname{supp} g_k$ for $k = 1, 2$, and $\xi_0 = \xi_1 + \xi_2 \in \operatorname{supp} f$ we have

$$|\xi_{1,2}| + |\xi_{1,3}| + |\xi_{2,2}| + |\xi_{2,3}| + |\xi_{0,2}| + |\xi_{0,3}| \lesssim 1. \tag{3.22}$$

This shows that $|\xi_{1,1} + \xi_{2,1}| = |\xi_{0,1}| \sim N$, $|\xi_{1,1}|, |\xi_{2,1}| \sim N_1$.

In the following, we use almost orthogonality methods to further localize all functions to smaller pieces, for which the claim is trivial.

By decomposing f, g_1, g_2 into $\sim L, L_1, L_2$ pieces, respectively, and applying the Cauchy–Schwarz inequality, it suffices to prove

$$\left| \int g_1(\xi_1, \tau_1) g_2(\xi_2, \tau_2) f(\xi_1 + \xi_2, \tau_1 + \tau_2)\, d\xi_1\, d\xi_2\, d\tau_1\, d\tau_2 \right| \\ \lesssim N_1^{-\frac{1}{2}} \|g_1\|_{L^2} \|g_2\|_{L^2} \|f\|_{L^2}, \tag{3.23}$$

where f is now supported in $c \leqslant \tau - |\xi| \leqslant c + 1$ and g_k is supported in $c_k \leqslant \tau_k - |\xi_k|^2 \leqslant c_k + 1$. Therefore, with respect to the τ variable, f is supported in an interval of length $\sim N$. Using orthogonality, we can further localize g_k with respect to the second variable τ_k to intervals of length $\sim N$, $k = 1, 2$. In turn this implies that the spatial frequencies ξ_k can be localized further to annuli of width $\sim N N_1^{-1} \leqslant 1$. In light of (3.22) we can strengthen the localization of g_k with respect to ξ_k to cubes of side-length ~ 1. As a consequence, we also improve the localization of the ξ-support of f to cubes of size ~ 1, which then also allows to localize f with respect to τ to intervals of length ~ 1. Now, we repeat the above procedure: We can further localize g_k with respect to τ_k to intervals of length ~ 1, which also implies a better localization for g_k with respect to ξ_k to annuli of width $\sim N_1^{-1}$.

In summary, we have reduced the problem to the case when the volume of the supports of g_1 and g_2 is $\sim N_1^{-1}$ which then trivially gives (3.23) by virtue of the Cauchy–Schwarz inequality. □

Next, we summarize the previous two results in the following corollary, which settles the high–high to low interactions with low modulation.

Corollary 3.6 *(High–high to low interactions, low modulation). Assume that* $f, g_1, g_2 \in L^2$ *with* $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ *and*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

where N, N_1, N_2 *and* L, L_1, L_2 *satisfy*

$$1 \ll N \lesssim N_1 \sim N_2, \qquad L_1, L_2, L \lesssim N_1^2.$$

Then, the following estimate holds

$$\bigl|I(f, g_1, g_2)\bigr| \lesssim (L_1 L_2 L)^{\frac{1}{2}} N_1^{-\frac{1}{2}} \log N_1. \tag{3.24}$$

Proof. It suffices to consider non-negative f, g_1, g_2. We choose a threshold $M = CN_1$ such that for $A < M$ Proposition 3.4 respectively for $A = M$ Proposition 3.5 is applicable, and decompose

$$\begin{aligned}\bigl|I(f, g_1, g_2)\bigr| \leqslant{}& \sum_{A=1}^{M-1} \sum_{\alpha(j_1,j_2)\sim A^{-1}} I\bigl(f, Q_A^{j_1} g_1, Q_A^{j_2} g_2\bigr)\\ &+ \sum_{\alpha(j_1,j_2)\lesssim M^{-1}} I\bigl(f, Q_M^{j_1} g_1, Q_M^{j_2} g_2\bigr).\end{aligned}$$

Concerning the first sum, we use (3.14) for fixed A and obtain

$$\begin{aligned}\sum_{\alpha(j_1,j_2)\sim A^{-1}} I\bigl(f, Q_A^{j_1} g_1, Q_A^{j_2} g_2\bigr) &\lesssim \frac{(L_1 L_2 L)^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \sum_{\alpha(j_1,j_2)\sim A^{-1}} \bigl\|Q_A^{j_1} g_1\bigr\|_{L^2} \bigl\|Q_A^{j_2} g_2\bigr\|_{L^2}\\ &\lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}},\end{aligned}$$

where, in the last step, we use Cauchy–Schwarz inequality. Concerning the second sum, we use (3.21) for fixed A and obtain the same bound as above.

Dyadic summation with respect to A introduces the additional factor $\log N_1$, which leads to (3.24). □

The case of high–high to low interactions with high modulation is covered by the following proposition.

Proposition 3.7 *(High–high to low interactions, high modulation). Assume that $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

where N, N_1, N_2 and L, L_1, L_2 satisfy

$$1 \leqslant N \lesssim N_1 \sim N_2, \qquad N_1^2 \lesssim \max\{L, L_1, L_2\}.$$

Then, the following estimate holds

$$\bigl|I(f, g_1, g_2)\bigr| \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \bigl(\max\{L, L_1, L_2\} N_1^{-2}\bigr)^{-\frac{1}{2}}. \tag{3.25}$$

Proof. *Case (a):* $L = \max\{L, L_1, L_2\}$: We use the Cauchy–Schwarz inequality and (3.11).

Case (b): $L_1 = \max\{L, L_1, L_2\}$ or $L_2 = \max\{L, L_1, L_2\}$: We use the Cauchy–Schwarz inequality and (3.13). □

The next proposition covers the case of low–high interactions.

Proposition 3.8 *(Low–high interactions). Let $f, g_1, g_2 \in L^2$ be functions with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

with $1 \leqslant N_1 \ll N_2$.

(i) *If $L_2 \ll N_2^2$, then we have*

$$\left|I(f, g_1, g_2)\right| \lesssim N_1 N_2^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \max(L, L_1, L_2)^{-\frac{1}{2}}. \tag{3.26}$$

(ii) *If $L_2 \gtrsim N_2^2$, then we have*

$$\left|I(f, g_1, g_2)\right| \lesssim N_1^{\frac{1}{2}} \min\{L, L_1\}^{\frac{1}{2}} \min\left\{N_1^2, \max\{L, L_1\}\right\}^{\frac{1}{2}}. \tag{3.27}$$

Proof. The integral vanishes unless $N_2 \sim N$ and

$$\max\{L, L_1, L_2\} \gtrsim \left||\xi_1|^2 - |\xi_2|^2 \pm |\xi_1 - \xi_2|\right| \gtrsim N_2^2. \tag{3.28}$$

We split the proof into two cases:

Case (a): $L_2 \ll N_2^2$.

Subcase (i): $L = \max\{L, L_1, L_2\}$. The bilinear L^2 estimate (3.11) yields

$$\left|I(f, g_1, g_2)\right| \lesssim \|f\|_{L^2} \left\|\mathcal{F}^{-1} g_1 \overline{\mathcal{F}^{-1} g_2}\right\|_{L^2} \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} N_1 N_2^{-\frac{1}{2}}.$$

Subcase (ii): $L_1 = \max\{L, L_1, L_2\}$. Since g_1 is localized to a cube of sidelength N_1 with respect to the ξ_1 variable, by almost orthogonality the estimate reduces to the case when f and g_2 are similarly localized to cubes of sidelength N_1. Then we use the bilinear L^2 estimate (3.12) with $d = N_1$ to obtain

$$\left|I(f, g_1, g_2)\right| \lesssim \|g_1\|_{L^2} \left\|\mathcal{F}^{-1} f \mathcal{F}^{-1} g_2\right\|_{L^2} \lesssim L^{\frac{1}{2}} L_2^{\frac{1}{2}} N_1 N_2^{-\frac{1}{2}}.$$

This finishes the proof of (3.26).

Case (b): $L_2 \gtrsim N_2^2$. Again, since g_1 is localized to a cube of sidelength N_1 with respect to the ξ variable, the estimate reduces to the case when f and g_2 are localized to cubes of sidelength N_1 with respect to the ξ variables.

Subcase (i): $L \leqslant L_1$ and $N_1^2 \leqslant \max\{L, L_1\}$. The volume of the support of f is $\lesssim N_1^3 L$, and we estimate

$$\left|I(f, g_1, g_2)\right| \lesssim \|f\|_{L^1} \|g_1\|_{L^2} \|g_2\|_{L^2} \lesssim N_1^{\frac{3}{2}} L^{\frac{1}{2}}.$$

Subcase (ii): $L_1 < L$ and $N_1^2 \leqslant \max\{L, L_1\}$. The volume of the support of g_1 is $\lesssim N_1^3 L_1$, and we estimate

$$\left|I(f, g_1, g_2)\right| \lesssim \|f\|_{L^2} \|g_1\|_{L^1} \|g_2\|_{L^2} \lesssim N_1^{\frac{3}{2}} L_1^{\frac{1}{2}}.$$

Subcase (iii): $N_1^2 > \max\{L, L_1\}$. The Cauchy–Schwarz inequality and the bilinear L^2 estimate (3.12) yield

$$\left|I(f, g_1, g_2)\right| \lesssim \|\mathcal{F}^{-1} f \mathcal{F}^{-1} g_1\|_{L^2} \|g_2\|_{L^2} \lesssim N_1^{\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}},$$

which finishes the proof of (3.27). □

Finally, we deal with the case where the wave frequency is very small.

Proposition 3.9 *(Very small wave frequency). Assume that* $f, g_1, g_2 \in L^2$ *with* $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ *such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \qquad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

and assume that $N \lesssim 1$. *Then,*

$$\left|I(f, g_1, g_2)\right| \lesssim \min(L, L_1, L_2)^{\frac{1}{2}}. \tag{3.29}$$

Proof. Using orthogonality we reduce the problem to the case when both g_1 and g_2 are supported in cubes of size ~ 1 with respect to the ξ_k variables. Then, the volume of the support of f is L, while the volume of the support of g_k is L_k. If $L = \min(L, L_1, L_2)$, then by using the trivial estimate

$$\left|I(f, g_1, g_2)\right| \lesssim \|f\|_{L^1} \|g_1\|_{L^2} \|g_2\|_{L^2} \lesssim L^{\frac{1}{2}}$$

the claim follows. If $L_1 = \min(L, L_1, L_2)$ then we obtain

$$\left|I(f, g_1, g_2)\right| \lesssim \|f\|_{L^2} \|g_1\|_{L^1} \|g_2\|_{L^2} \lesssim L_1^{\frac{1}{2}}.$$

The case $L_2 = \min(L, L_1, L_2)$ follows in a similar manner. □

3.3. *Proof of Proposition 3.2*

We prove parts (i) and (ii) at the same time. We focus on establishing (3.8) and (3.9) as stated in part (i). Then, at any step we show that we can improve the corresponding estimate by using the $X_{s,b,\infty}$ norm instead of $X_{s,\frac{1}{2},\infty}$ norm on of the terms involved in the estimate, where b is a parameter which depends on s and σ. The conditions on b will accumulate in several steps but one has to keep in mind that $b < \frac{1}{2}$ is the starting condition and it will not be repeated.

Proof of Proposition 3.2. By definition of the norms it is enough to consider functions with non-negative Fourier transform. We dyadically decompose

$$u_k = \sum_{N_k, L_k \geqslant 1} S_{L_k} P_{N_k} u_k, \qquad v = \sum_{N, L \geqslant 1} W_L^{\pm} P_N v.$$

Setting $g_k^{L_k, N_k} = \mathcal{F} S_{L_k} P_{N_k} u_k$ and $f^{L,N} = \mathcal{F} W_L^{\pm} P_N v$, we observe

$$I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2) = \sum_{N, N_1, N_2 \geqslant 1} \sum_{L, L_1, L_2 \geqslant 1} I\big(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2}\big).$$

Case (a): high–high–low interactions, i.e. $N_1 \sim N_2 \gtrsim N \gg 1$. Using (3.24) and (3.25) it follows that

$$\begin{aligned}
&\sum_{L,L_1,L_2 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right| \\
&\quad\lesssim N_1^{-\frac{1}{2}} \log N_1 \sum_{L,L_1,L_2 \leqslant N_1^2} L^{\frac{1}{2}} \left\| f^{L,N}\right\|_{L^2} L_1^{\frac{1}{2}} \left\| g_1^{L_1,N_1}\right\|_{L^2} L_2^{\frac{1}{2}} \left\| g_2^{L_2,N_2}\right\|_{L^2} \\
&\qquad + N_1^{-\frac{1}{2}} \sum_{\max\{L,L_1,L_2\} > N_1^2} \frac{N_1}{\max\{L, L_1, L_2\}^{\frac{1}{2}}} L^{\frac{1}{2}} \left\| f^{L,N}\right\|_{L^2} L_1^{\frac{1}{2}} \left\| g_1^{L_1,N_1}\right\|_{L^2} L_2^{\frac{1}{2}} \left\| g_2^{L_2,N_2}\right\|_{L^2} \\
&\quad\lesssim N_1^{-\frac{1}{2}} (\log N_1)^4 \|P_N v\|_{X^{w\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}.
\end{aligned}$$

A straightforward modification also shows the bound

$$\lesssim N_1^{\frac{1}{2}-2b} (\log N_1)^4 \|P_N v\|_{X^{w\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,b,\infty}}.$$

In order to prove (3.8) we perform the summation with respect to $1 \ll N \leqslant N_1 \sim N_2$ and obtain

$$\begin{aligned}
&\sum_{1 \ll N \leqslant N_1 \sim N_2} N^{-\sigma} N_1^{-\frac{1}{2}} (\log N_1)^4 \|P_N v\|_{X^{w\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{-s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}} \\
&\quad\lesssim \|v\|_{X^{w\pm}_{\sigma,\frac{1}{2},\infty}} \sum_{1 \ll N_1 \sim N_2} \|P_{N_1} u_1\|_{X^S_{-s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}} \\
&\quad\lesssim \|v\|_{X^{w\pm}_{\sigma,\frac{1}{2},\infty}} \|u_1\|_{X^S_{-s,\frac{1}{2},\infty}} \|u_2\|_{X^S_{s,\frac{1}{2},\infty}},
\end{aligned}$$

where we have used that $\sigma > -\frac{1}{2}$. If we choose b such that $\frac{1}{2} - 2b - \sigma < 0$, then we also obtain

$$\sum_{1 \ll N \leqslant N_1 \sim N_2} \left| I\left(f^N, g_1^{N_1}, g_2^{N_2}\right)\right| \lesssim \|v\|_{X^{w\pm}_{\sigma,\frac{1}{2},\infty}} \|u_1\|_{X^S_{-s,\frac{1}{2},\infty}} \|u_2\|_{X^S_{s,b,\infty}}.$$

For proving (3.9) in this case, we perform the summation as follows:

$$\begin{aligned}
&\sum_{1 \ll N \leqslant N_1 \sim N_2} \left(\frac{N}{N_1}\right)^{1+\sigma} N_1^{\frac{1}{2}+\sigma-2s} (\log N_1)^4 \|P_N v\|_{X^{w\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}} \\
&\quad\lesssim \|v\|_{X^{w\pm}_{-1-\sigma,\frac{1}{2},\infty}} \sum_{1 \ll N_1 \sim N_2} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}} \\
&\quad\lesssim \|v\|_{X^{w\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}},
\end{aligned}$$

where we have used that $\sigma - 2s < -\frac{1}{2}$. By picking b such that $\frac{3}{2} - 2b + \sigma - 2s < 0$, we also obtain

$$\sum_{1 \ll N \leqslant N_1 \sim N_2} \left| I\left(f^N, g_1^{N_1}, g_2^{N_2}\right)\right| \lesssim \|v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|u_2\|_{X^S_{s,b,\infty}}.$$

Case (b): very small wave frequency, i.e. $N \lesssim 1$. In this case, either $N_1 \sim N_2$ or $N, N_1, N_2 \lesssim 1$. We use (3.29) and obtain

$$\begin{aligned}
&\sum_{L,L_1,L_2 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right| \\
&\quad \lesssim \sum_{1 \leqslant L,L_1,L_2} \left(\frac{\min\{L, L_1, L_2\}}{L L_1 L_2}\right)^{\frac{1}{2}} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}} \\
&\quad \lesssim \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}.
\end{aligned}$$

A similar argument shows

$$\sum_{L,L_1,L_2 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right| \lesssim \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},1}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,b,\infty}},$$

provided that $b > 0$. (3.8) and (3.9) and their counterpart in (ii) follow from these estimates since $N_1 \sim N_2$ or $N_1, N_2 \lesssim 1$.

Case (c): high–low interactions, i.e. $N_1 \ll N_2$ or $N_1 \gg N_2$. We focus on the case $N_1 \ll N_2$, the other one being similar. Since we apply Proposition 3.8 we need to differentiate between the cases $L_2 \ll N_2^2$ and $L_2 \gtrsim N_2^2$. In the first case, by (3.26) and the observation (3.28) which implies that $\max(L, L_1) \gtrsim N_2^2$ for non-vanishing interactions, we have

$$\begin{aligned}
&\sum_{L_2 \ll N_2^2} \sum_{L,L_1 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right| \\
&\quad \lesssim N_1 N_2^{-\frac{3}{2}} \sum_{L_2 \ll N_2^2} \sum_{L,L_1 \geqslant 1} \left\langle \frac{\max(L, L_1)}{N_2^2} \right\rangle^{-\frac{1}{2}} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}} \\
&\quad \lesssim N_1 N_2^{-\frac{3}{2}} (\ln N_2)^2 \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}.
\end{aligned}$$

By the same reasoning we also have

$$\begin{aligned}
&\sum_{L_2 \ll N_2^2} \sum_{L,L_1 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right| \\
&\quad \lesssim N_1 N_2^{-\frac{1}{2}-2b} (\ln N_2)^2 \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,b,\infty}}.
\end{aligned}$$

In the case $L_2 \gtrsim N_2^2$ we use (3.27) and obtain

$$\sum_{L_2 \gtrsim N_2^2} \sum_{L, L_1 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right|$$
$$\lesssim N_1^{\frac{1}{2}} \sum_{L_2 \gtrsim N_2^2} \sum_{1 \leqslant L, L_1 \leqslant N_1^2} L_2^{-\frac{1}{2}} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}$$
$$+ N_1^{\frac{3}{2}} \sum_{L_2 \gtrsim N_2^2} \sum_{\max\{L, L_1\} > N_1^2} \left(\max\{L, L_1\} L_2\right)^{-\frac{1}{2}} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}$$
$$\lesssim N_1^{\frac{1}{2}} (\ln N_1)^2 N_2^{-1} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}.$$

In a similar manner we obtain

$$\sum_{L_2 \gtrsim N_2^2} \sum_{L, L_1 \geqslant 1} \left| I\left(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}\right)\right|$$
$$\lesssim N_1^{\frac{3}{2}-2b} (\ln N_1)^2 N_2^{-1} \|P_N v\|_{X^{W\pm}_{0,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{0,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{0,\frac{1}{2},\infty}}.$$

For proving (3.8) we estimate the above term in the worst case in which we place the low Schrödinger frequency in the space with positive Sobolev regularity and the high Schrödinger frequency in the space with negative Sobolev regularity. It is obvious that the other case gives better estimates. From the above inequalities we deduce

$$\sum_{N_1 \ll N \sim N_2} \left| I\left(f^N, g_1^{N_1}, g_2^{N_2}\right)\right|$$
$$\lesssim \sum_{N_1 \ll N \sim N_2} N_1^{-s+\frac{1}{2}} N^{s-1-\sigma} (\ln N_1)^2 \|P_N v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{-s,\frac{1}{2},\infty}}.$$

If $s \leqslant \frac{1}{2}$, then we can bound the above sum by

$$\|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{-\frac{1}{2}-\sigma} (\ln N)^3 \|P_N v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{-s,\frac{1}{2},\infty}}$$
$$\lesssim \|P_N v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{-s,\frac{1}{2},\infty}}$$

where we have used that $\sigma > -\frac{1}{2}$.

If $s > \frac{1}{2}$, then we can bound the above sum by

$$\|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{s-1-\sigma} \|P_N v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{-s,\frac{1}{2},\infty}}$$
$$\lesssim \|P_N v\|_{X^{W\pm}_{\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{-s,\frac{1}{2},\infty}}$$

where we have used that $s \leqslant 1+\sigma$. As noted earlier, it is obvious that in the case $N_2 \ll N_1 \sim N$ the above estimate is easier. With similar arguments we can verify the counterpart in (ii) of these estimates, but we omit the details.

Concerning (3.9) we proceed as follows:

$$\sum_{N_1 \ll N \sim N_2} \left| I\left(f^N, g_1^{N_1}, g_2^{N_2}\right)\right|$$
$$\lesssim \sum_{N_1 \ll N \sim N_2} N_1^{-s+\frac{1}{2}} N^{-s+\sigma} (\ln N_1)^2 \|P_N v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}}.$$

If $s \leqslant \frac{1}{2}$, then we can bound the above sum by

$$\|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{-2s+\frac{1}{2}+\sigma} (\ln N)^2 \|P_N v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}}$$
$$\lesssim \|P_N v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}}$$

where we have used that $\sigma + \frac{1}{2} < 2s$.

If $s > \frac{1}{2}$, then we can bound the above sum by

$$\|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{-s+\sigma} \|P_N v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}}$$
$$\lesssim \|P_N v\|_{X^{W\pm}_{-1-\sigma,\frac{1}{2},\infty}} \|P_{N_1} u_1\|_{X^S_{s,\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^S_{s,\frac{1}{2},\infty}}$$

where we have used that $\sigma \leqslant s$.

It is an easy exercise to verify the counterpart in (ii) of these estimates. This concludes the proof of Proposition 3.2. □

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